JOURNAL OF APPROXIMATION THEORY 29, 323–335 (1980)

Asymptotic Properties of Powers of Bernstein Operators

JOSEF NAGEL

Fachbereich 6—Mathematik, Universität Essen—Gesamthochschule, Postfach 6843, D-4300 Essen 1, West Germany

Communicated by Lothar Collatz

Received March 18, 1979

DEDICATED TO THE MEMORY OF P. TURÁN

1. INTRODUCTION

The Bernstein operators B_n: $\mathscr{C}[0, 1] \to \mathscr{C}[0, 1]$ are defined by

$$B_n(f; x) = \sum_{k=0}^{n} \binom{n}{k} x^k(1-x)^{n-k} f\left(\frac{k}{n}\right) \qquad (f \in \mathscr{C}[0, 1]; x \in [0, 1]).$$

Throughout this paper, let $(k_n)_{n\in\mathbb{N}}$ be a sequence of natural numbers. We investigate the limit behaviour of the sequence $(B_n^{k_n})_{n\in\mathbb{N}}$ of powers of Bernstein operators. Using Hilbert space methods, we give an explicit formula for $\lim_{n\to\infty} B_n^{k_n}f$, provided $\lim_{n\to\infty}(k_n/n)$ exists and f is smooth, i.e., $f' \in \mathscr{L}_2[0, 1]$. The limits of the eigenfunctions of B_n prove to be the indefinite integrals of Legendre polynomials. For the cases $\lim_{n\to\infty}(k_n/n) = 0$ and $\lim_{n\to\infty}(k_n/n) = \infty$, we shall give theorems of Voronovskaja type. Since our work depends fundamentally on a work of Kelisky and Rivlin [2], we summarize their main results in the next section.

We shall use the following notation: $\mathbb{N}$ denotes the set of all natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The space of all continuous real-valued functions on the closed interval [0, 1], resp. k-times continuously differentiable real-valued functions on [0, 1], is denoted by $\mathscr{C}[0, 1]$, resp. $\mathscr{C}^{(k)}[0, 1]$. For each $s \in \mathbb{N}$, $\mathscr{P}_s$ is the space of all real polynomials of degree s and $\mathscr{P}_{s0}$ the subspace of all $p \in \mathscr{P}_s$ with $p(0) = 0$; similarly $\mathscr{P}$ is the space of all real polynomials and $\mathscr{P}_0$ the subspace of all $p \in \mathscr{P}$ with $p(0) = 0$. We consider these polynomial spaces as subspaces of $\mathscr{C}[0, 1]$. For $s \in \mathbb{N}_0$, e_s is the monomial $e_s(x) = x^s$. $\|\cdot\|_\infty$ will denote the supremum norm on $\mathscr{C}[0, 1]$ and $\|\cdot\|_p$ the norm on the function spaces $\mathscr{L}_p[0, 1]$ ($p \geqslant 1$). The norms of operators on these spaces will be denoted by the same symbols.

0021-9045/80/080323-13$02.00/0

2. Main Results of Kelisky and Rivlin [2]

Let $s \in \mathbb{N}$ be fixed. Since $B_n p \in \mathscr{P}_s$ for all $p \in \mathscr{P}_s$ and $B_n(f; 0) = f(0)$ for all $f \in \mathscr{C}[0, 1]$, we can interpret each B_n as a linear map $B_n\colon \mathscr{P}_{s0} \to \mathscr{P}_{s0}$. For $j \in \{1,..., s\}$ one obtains $B_n(e_j\,; x) = a_{1j}(n)x + a_{2j}(n)\, x^2 + \cdots + a_{jj}(n)\, x^j$ with

$$a_{ij}(n) = \pi_i(n)\, n^{i-j} \sigma_j^{\,i} \qquad \text{for} \quad i \leqslant j,$$
$$= 0 \qquad \text{for} \quad i > j,$$

$$\pi_i(n) = 1 \qquad \text{for} \quad i = 1,$$
$$= \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) \cdots \left(1 - \frac{i-1}{n}\right) \qquad \text{for} \quad i > 1,$$

$$\sigma_j^{\,i} = \frac{(-1)^i}{i!} \sum_{k=1}^{i} (-1)^k \binom{i}{k} k^j \qquad (i, j \in \{1,..., s\})$$

(Stirling numbers of the second kind).

The associated $s \times s$ matrix $A(n)$ with elements $a_{ij}(n)$ is upper triangular. In the following, we shall only consider indices n with $n \geqslant s$. Then $A(n)$ possesses the eigenvalues $\pi_i(n)$ $(i = 1,..., s)$ and can be diagonalized. Let $V(n)$—with elements $v_{ij}(n)$—be the $s \times s$ matrix of eigenvectors with the standardization $v_{ii}(n) = 1$ $(i = 1,..., s)$. Also $V(n)$ is upper triangular. Kelisky and Rivlin showed that the $V(n)$ converge, and calculated the limit matrix. A slight transformation of the original Kelisky and Rivlin formula yields

$$\lim_{n\to\infty} v_{ij}(n) = (-1)^{i+j} \frac{\binom{j-1}{i-1}\binom{j}{i}}{\binom{2j-2}{j-i}} \qquad (i, j \in \{1,..., s\},\ i \leqslant j).$$

For each $n \geqslant s$, to the eigenvectors $(v_{1j}(n), v_{2j}(n),..., v_{sj}(n))^t$ there correspond polynomials $p_{jn} \in \mathscr{P}_{j0}$ $(j = 1,..., s)$ which are eigenfunctions of B_n and for which therefore the following relation holds:

$$B_n p_{jn} = p_{1n} \qquad \text{for} \quad j = 1,$$
$$= \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) \cdots \left(1 - \frac{j-1}{n}\right) p_{jn} \qquad \text{for} \quad j = 2,..., s. \tag{1}$$

For each $j \in \{1,..., s\}$, the coefficients of p_{jn} converge (as $n \to \infty$) to the corresponding coefficients of the polynomial p_j, where

$$p_j(x) = \sum_{i=1}^{j} (-1)^{i+j} \frac{\binom{j-1}{i-1}\binom{j}{i}}{\binom{2j-2}{j-i}} x^i \tag{2}$$

(in what follows, the expression "coefficientwise convergent" and the denotation "$\lim_{n\to\infty} p_{jn} =^c p_j$" will be used).

3. Asymptotic Behaviour of $B_n^{k_n}p$ for Polynomials p

Throughout this section, let $p \in \mathscr{P}_0$ be a given polynomial, say $p(x) = \sum_{i=1}^{s} a_i x^i$, of degree s. There are unique representations $p = \sum_{j=1}^{s} b_{jn} p_{jn}$ and $p = \sum_{j=1}^{s} b_j p_j$ with coefficients b_{jn} and b_j. Again for all occurring indices n, we assume the restriction $n \geqslant s$. The application of $B_n^{k_n}$ yields

$$\begin{aligned} B_n^{k_n}p &= b_{1n}p_{1n} + b_{2n}\left(1-\frac{1}{n}\right)^{k_n} p_{2n} + b_{3n}\left(1-\frac{1}{n}\right)^{k_n}\left(1-\frac{2}{n}\right)^{k_n} p_{3n} \\ &\quad + \cdots + b_{sn}\left(1-\frac{1}{n}\right)^{k_n}\left(1-\frac{2}{n}\right)^{k_n} \cdots \left(1-\frac{s-1}{n}\right)^{k_n} p_{sn}. \end{aligned} \tag{3}$$

For our further investigation we need two lemmas.

Lemma 1.

(i) $b_{1n} = b_1 = p(1)$ *and* $p_{1n} = p_1 = e_1$ *for each* $n \geqslant s$;

(ii) $\lim_{n\to\infty} b_{jn} = b_j$ *for each* $j \in \{2,..., s\}$.

Proof. (i) Since we chose the standardization $v_{ii}(n) = 1$, there holds $p_{1n} = p_1 = e_1$. From (1) and the relation $B_n(f; 1) = f(1)$ for all $f \in \mathscr{C}[0, 1]$, it follows that $p_{jn}(1) = p_j(1) = 0$ for $j \in \{2,..., s\}$, and therefore $b_{1n} = \sum_{j=1}^{s} b_{jn}p_{jn}(1) = p(1) = \sum_{j=1}^{s} b_j p_j(1) = b_1$.

(ii) Since p_j and p_{jn} have leading coefficients 1, $p = \sum_{j=1}^{s} b_j p_j = \sum_{j=1}^{s} b_{jn} p_{jn}$ implies $b_s = b_{sn}$ for all $n \geqslant s$. Now let be $k \in \{1,..., s-1\}$, and suppose the convergence $\lim_{n\to\infty} b_{jn} = b_j$ is known for all $j \in \{k+1,..., s\}$. To establish $\lim_{n\to\infty} b_{kn} = b_k$ consider the relation $\sum_{j=1}^{k} (b_j p_j - b_{jn} p_{jn}) = \sum_{j=k+1}^{s} (b_{jn} p_{jn} - b_j p_j)$. The right side converges coefficientwise to the zero polynomial. Hence, in particular the leading coefficient on the left side converges to zero, i.e., $\lim_{n\to\infty}(b_k - b_{kn}) = 0$. ∎

LEMMA 2. *Suppose $(k_n)_{n\in\mathbb{N}}$ is a sequence of natural numbers with $\lim_{n\to\infty}(k_n/n) = 0$. Then for each $l \in \mathbb{N}$, $l \geqslant 2$, we have*

$$\lim_{n\to\infty} \frac{n}{k_n}\left\{\left(1-\frac{1}{n}\right)^{k_n}\left(1-\frac{2}{n}\right)^{k_n}\cdots\left(1-\frac{l-1}{n}\right)^{k_n}-1\right\} = -\binom{l}{2}.$$

Proof. The assertion can be reduced to

$$\lim_{n\to\infty} \frac{n}{k_n}\left\{\left(1-\frac{m}{n}\right)^{k_n}-1\right\} = -m \qquad \text{for each} \quad m\in\mathbb{N},$$

which is verified by using standard techniques of mathematical analysis. The details are left to the reader. ∎

The following proposition is a straightforward consequence of (1), (3), and Lemmas 1 and 2.

PROPOSITION 1. *Let p be the given polynomial and $(k_n)_{n\in\mathbb{N}}$ a sequence of natural numbers.*

(i) *In the case $\lim_{n\to\infty}(k_n/n) = 0$ we have*

$$\lim_{n\to\infty} B_n^{k_n} p \overset{c}{=} b_1p_1 + b_2p_2 + \cdots + b_sp_s = p. \tag{4}$$

As to the degree of approximation we obtain

$$\lim_{n\to\infty} \frac{n}{k_n}\{B_n^{k_n}p - p\} = \lim_{n\to\infty}\frac{n}{k_n}\left\{\sum_{j=1}^{s} b_{jn}(B_n^{k_n}p_{jn} - p_{jn})\right\}$$

$$\overset{c}{=} -\sum_{j=2}^{s}\binom{j}{2} b_jp_j\,. \tag{5}$$

(ii) *In the case $\lim_{n\to\infty}(k_n/n) = \infty$ we have*

$$\lim_{n\to\infty} B_n^{k_n}p \overset{c}{=} b_1p_1 = p(1)\,e_1\,. \tag{6}$$

As to the degree of approximation we obtain

$$\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^{-k_n}\{B_n^{k_n}p - b_1p_1\} \overset{c}{=} b_2p_2\,. \tag{7}$$

(iii) *In the case $\lim_{n\to\infty}(k_n/n) = q \in (0,\infty)$ using the abbreviations $E_j := e^{-j(j-1)/2}$ $(j = 1,\ldots, s)$ we have*

$$\lim_{n\to\infty} B_n^{k_n}p \overset{c}{=} b_1p_1 + E_2{}^q b_2p_2 + \cdots + E_s{}^q b_sp_s\,. \tag{8}$$

In this case a simple result concerning the degree of approximation seems to be impossible.

In the next section we answer the question, how do the coefficients b_j depend on the given function p?

4. The Associated Hilbert Space $\mathscr{H}_B$

Let $\mathscr{H}_B$ denote the space of all absolutely continuous real-valued functions on $[0, 1]$ with $f(0) = 0$ and $f' \in \mathscr{L}_2[0, 1]$. For $f, g \in \mathscr{H}_B$ define $\langle f, g\rangle := \int_0^1 f'(t)\, g'(t)\, dt$ and $\|f\|_B := (\int_0^1 f'(t)^2\, dt)^{1/2}$. Obviously $\langle \cdot, \cdot \rangle$ is an inner product on $\mathscr{H}_B$, and thereby $\mathscr{H}_B$ becomes a real Hilbert space with norm $\|\cdot\|_B$ ($\mathscr{H}_B$ is closely related to a certain Sobolev space). Using Hölder's inequality we get

$$|f(x)| \leqslant \int_0^x |f'(t)|\, dt \leqslant \|f\|_B\, x^{1/2} \qquad (x \in [0, 1]) \tag{9}$$

and therefore $\|f\|_\infty \leqslant \|f\|_B$ for all $f \in \mathscr{H}_B$. Hence on $\mathscr{H}_B$ the Hilbert space topology is finer than the topology of uniform convergence. On $\mathscr{H}_B$ the norms $\|f\|_B = \|f'\|_2$ and $\|f\|_2^1 := \|f\|_2 + \|f'\|_2$ are equivalent; more exactly we have $\frac{1}{2}\|f\|_2^1 \leqslant \|f\|_B \leqslant \|f\|_2^1$ for all $f \in \mathscr{H}_B$, a simple conclusion from (9). Since the polynomials are dense in $\mathscr{L}_2[0, 1]$, $\mathscr{P}_0$ is a dense subset of the Hilbert space $\mathscr{H}_B$.

Now for $j \in \mathbb{N}$ we define polynomials f_j by

$$f_j := \frac{1}{(2j-1)^{1/2}} \binom{2j-1}{j} p_j, \tag{10}$$

where p_j as in (2). For $j \in \mathbb{N}_0$ let g_j denote the Legendre polynomial of degree j on the interval $[0, 1]$.

Proposition 2.

(i) $\{f_j \mid j \in \mathbb{N}\}$ *is a complete orthonormal set in* $\mathscr{H}_B$;

(ii) $f_j(x) = \int_0^x g_{j-1}(t)dt$ *for all* $j \in \mathbb{N}$.

Proof. For $j \geqslant 2$, we use the representation

$$f_j(x) = (-1)^{j+1} \frac{(2j-1)^{1/2}}{(j-1)!} h_j^{(j-2)}(x) \tag{11}$$

with the auxiliary function $h_j(x) = (x(1-x))^{j-1}$. Then (i) ensues by standard arguments, and (ii) is obvious, when we emphasize (i) and the definition of Legendre polynomials. ∎

Thus each $f \in \mathscr{H}_B$ admits an expansion $f = \sum_{j=1}^{\infty} \langle f, f_j \rangle f_j$, and by the above remarks we infer that this expansion is also valid with respect to the supremum norm on $\mathscr{C}[0, 1]$. For polynomials $p \in \mathscr{P}_{s0}$, from (10) and the representation $p = \sum_{j=1}^{s} \langle p, f_j \rangle f_j$ we get an explicit formula for the coefficients b_j of Section 3:

$$b_j = \binom{2j-1}{j}^2 \frac{1}{2j-1} \int_0^1 p_j'(t)\, p'(t)\, dt. \tag{12}$$

Inserting in Proposition 1 yields new formulations for (5) resp. (7), which are marked with (5′) resp. (7′).

In the case $\lim_{n\to\infty}(k_n/n) = 0$, we have

$$\lim_{n\to\infty} \frac{n}{k_n} \{B_n^{k_n} p - p\} \overset{c}{=} A_0 p, \tag{5′}$$

where

$$A_0(p; x) := \tfrac{1}{2} x(1-x)\, p''(x).$$

In the case $\lim_{n\to\infty}(k_n/n) = \infty$, we have

$$\lim_{n\to\infty} \left(1 - \frac{1}{n}\right)^{-k_n} \{B_n^{k_n} p - p(1)\, e_1\} \overset{c}{=} A_\infty p, \tag{7′}$$

where

$$A_\infty(p; x) := \frac{1}{2} x(1-x) \left\{6 \int_0^1 (1-2t)\, p'(t)\, dt\right\}.$$

(5′) follows from (10), (1) and the relation

$$x(1-x)\, h_j^{(j)}(x) = j(1-j)\, h_j^{(j-2)}(x) \qquad (j \geqslant 2),$$

whereas (7′) is immediate. Interpreting the Bernstein operators as linear operators $B_n: \mathscr{H}_B \to \mathscr{H}_B$, we are interested in the associated operator norm, which will be denoted by $\| B_n \|_B$.

PROPOSITION 3. *For all $k, n \in \mathbb{N}$, we have $\| B_n{}^k \|_B = 1$.*

Proof. We use the Kantorovič operators $P_n: \mathscr{L}_p[0, 1] \to \mathscr{L}_p[0, 1]$ ($p \geqslant 1$; $n \in \mathbb{N}_0$) defined by

$$P_n(f; x) := (n+1) \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} \int_{k/(n+1)}^{(k+1)/(n+1)} f(t)\, dt$$

$$(f \in \mathscr{L}_p[0, 1];\ x \in [0, 1]).$$

The following facts are known (cf. Lorentz [3, p. 30]): For $f \in \mathscr{L}_p[0, 1]$ with $F(x) := \int_0^x f(t)\,dt$, the relation

$$P_n(f; x) = \frac{d}{dx}(B_{n+1}(F; x)) \qquad (x \in [0, 1])$$

holds true, and for the operator norms we have

$$\| P_n \|_p \leqslant 1 \qquad \text{for all} \quad n \in \mathbb{N}_0, p \geqslant 1.$$

Thus for functions $f \in \mathscr{H}_B$ with $\|f\|_B = \|f'\|_2 \leqslant 1$, we get $\| B_n f\|_B = \|(B_n f)'\|_2 = \| P_{n-1} f'\|_2 \leqslant \|f'\|_2 = \|f\|_B \leqslant 1$, which implies $\| B_n{}^k \|_B \leqslant 1$. The converse inequality follows from $B_n e_1 = e_1$ and $\| e_1 \|_B = 1$. ∎

The main theorem of this section comprises a result about the covergence of the sequence $(B_n^{k_n} f)_{n \in \mathbb{N}}$ for functions $f \in \mathscr{H}_B$. As in Proposition 1, we shall use the abbreviations $E_j = e^{-j(j-1)/2}$ ($j \in \mathbb{N}$), and for $q = \infty$, we set $E_j{}^q := 1$ if $j = 1$ and $E_j{}^q := 0$ otherwise. In Theorem 1 all occurring convergence relations are to be understood with respect to the norm $\| \cdot \|_B$.

THEOREM 1.

For each $q \in [0, \infty]$,

$$\mathscr{B}_q f := \sum_{j=1}^{\infty} E_j{}^q \langle f, f_j \rangle f_j \qquad (f \in \mathscr{H}_B) \tag{13}$$

is a linear bounded operator, $\mathscr{B}_q \colon \mathscr{H}_B \to \mathscr{H}_B$, *with* $\| \mathscr{B}_q \|_B = 1$. *If* $(k_n)_{n \in \mathbb{N}}$ *is a sequence of natural numbers with* $\lim_{n\to\infty}(k_n/n) = q$, *then for each* $f \in \mathscr{H}_B$ *we have* $\lim_{n\to\infty} B_n^{k_n} f = \mathscr{B}_q f$.

Proof. Let $q \in [0, \infty]$ and let $(k_n)_{n \in \mathbb{N}}$ be a sequence of natural numbers with $\lim_{n\to\infty}(k_n/n) = q$. By Proposition 1, for each $f \in \mathscr{P}_0$, say, $f \in \mathscr{P}_{s0}$, the sequence $(B_n^{k_n} f)_{n \in \mathbb{N}}$ in $\mathscr{P}_{s0}$ is coefficient-wise convergent, and hence converges in the norm on $\mathscr{H}_B$. $\mathscr{P}_0$ is a dense subspace of $\mathscr{H}_B$, and on account of the above proposition, the norms $\| B_n^{k_n} \|_B$ are uniformly bounded by 1. Hence, the Banach–Steinhaus theorem ensures the existence of a linear bounded operator $\mathscr{B}_q \colon \mathscr{H}_B \to \mathscr{H}_B$ with $\| \mathscr{B}_q \| \leqslant 1$, such that $\lim_{n\to\infty} B_n^{k_n} f = \mathscr{B}_q f$. $\mathscr{B}_q e_1 = e_1$ and $\| e_1 \|_B = 1$ imply $\| \mathscr{B}_q \|_B = 1$. By virtue of the boundedness of $\mathscr{B}_q$, for each $f = \sum_{j=1}^{\infty} \langle f, f_j \rangle f_j \in \mathscr{H}_B$ we obtain $\mathscr{B}_q f = \sum_{j=1}^{\infty} \langle f, f_j \rangle \mathscr{B}_q f_j$, and (13) follows from Proposition 1. Finally we note that $\mathscr{B}_q f$ is independent of the special choice of the sequence $(k_n)_{n \in \mathbb{N}}$. ∎

About ten years ago, Karlin and Ziegler [1], Michelli [4], and Schnabl [6] gave the analogous theorem with respect to the uniform topology on $\mathscr{C}[0, 1]$. They proved the existence of linear operators $\mathscr{A}_q \colon \mathscr{C}[0, 1] \to \mathscr{C}[0, 1]$

($q \in [0, \infty]$) with $\|\mathscr{A}_q\|_\infty = 1$ such that the following holds: For each sequence $(k_n)_{n\in\mathbb{N}}$ of natural numbers with $\lim_{n\to\infty}(k_n/n) = q$ and for each $f \in \mathscr{C}[0, 1]$, $B_n^{k_n}f$ uniformly convergences to $\mathscr{A}_q f$ (as $n \to \infty$). But in contrast to (13), the operators $\mathscr{A}_q$ are not available (cf. another representation given by Karlin and Ziegler [1, p. 324]). Only for the cases $q = 0$ and $q = \infty$ one has $\mathscr{A}_0 = I$ (identity operator) by Korovkin's theorem and $\mathscr{A}_\infty = B_1$ by an analogous theorem due to Karlin and Ziegler ([1, Theorem 1]; cf. Sect. 5).

Illustrating this more exactly, let f be absolutely continuous and smooth, i.e., $f' \in \mathscr{L}_2[0, 1]$. Applying (13) to the function $g := f - B_1 f \in \mathscr{H}_B$, we get $\lim_{n\to\infty} B_n^{k_n}g = \sum_{j=2}^{\infty} \langle g, f_j\rangle E_j^q f_j$, which is valid with respect to the uniform topology on $\mathscr{C}[0, 1]$ as well. Since $B_n^{k_n}f = B_1 f + B_n^{k_n}g$, calculating the coefficients $\langle g, f_j\rangle$, we obtain with respect to the uniform topology on $\mathscr{C}[0, 1]$

$$\lim_{n\to\infty} B_n^{k_n}f = B_1 f + \sum_{j=2}^{\infty} E_j^q \left\{\int_0^1 [f(0) + t(f(1) - f(0)) - f(t)] f_j''(t)\, dt\right\} f_j, \tag{14}$$

i.e., both sides are uniformly convergent and coincide. Although the derivative f' does not appear on the right side of (14), the assumption $f' \in \mathscr{L}_2[0, 1]$ is still necessary for the validity of this equation, as the following consideration, for the case $q = 0$, will show.

For each $m \in \mathbb{N}$, let T_m be the linear operator $T_m: \mathscr{C}[0, 1] \to \mathscr{C}[0, 1]$ defined by $T_1 f := B_1 f$ and

$$T_m f := B_1 f + \sum_{j=2}^{m} \left\{\int_0^1 [f(0) + t(f(1) - f(0)) - f(t)] f_j''(t)\, dt\right\} f_j \quad (m \geqslant 2).$$

Obviously, each T_m is bounded with respect to the uniform topology on $\mathscr{C}[0, 1]$. By the above remarks, one readily shows that each T_m is a projection $T_m: \mathscr{C}[0, 1] \to \mathscr{P}_m$. Hence, by the Kharshiladze–Lozinski tkeorem we infer that there exists a function $f^* \in \mathscr{C}[0, 1]$ for which $T_m f^*$ is unbounded, i.e., for which the right side of (14) is unbounded, whereas on the left side $\lim_{n\to\infty} B_n^{k_n}f^* = f^*$ still holds true.

5. Quantitative Results for the Cases $\lim_{n\to\infty} (k_n/n) = 0$ and $\lim_{n\to\infty} (k_n/n) = \infty$

We first consider the case $\lim_{n\to\infty}(k_n/n) = 0$, which comprises the classical case $k_n = 1$. Applying Korovkin's theorem with the test set $\{e_0, e_1, e_2\}$, one obtains for all $f \in \mathscr{C}[0, 1]$ $\lim_{n\to\infty} B_n^{k_n}f = f$ uniformly on $[0, 1]$.

Estimates for the quality of the approximation $B_n^{k_n}f - f$ follow from general quantitative results for the approximation with linear positive operators, as derived in [5, Section 6]. For this purpose we need the defects of approximation with the functions of the test set, viz.,

$$d_0(x) := B_n^{k_n}(e_0\,; x) - e_0(x) = 0,$$

$$d_1(x) := B_n^{k_n}(e_1\,; x) - e_1(x) = 0,$$

$$d_2(x) := B_n^{k_n}(e_2\,; x) - e_2(x) = \left(1 - \left(1 - \frac{1}{n}\right)^{k_n}\right)(x - x^2).$$

From the proof of Lemma 2, for $d_2(x)$ we get

$$0 \leqslant d_2(x) \leqslant \frac{k_n}{n}\, x(1 - x) \leqslant \frac{k_n}{4n}.$$

Now employing Theorems 6.1 and 6.3 from [5], we obtain the following estimates:

Suppose $f \in \mathscr{C}^{(1)}[0, 1]$ and $f' \in \mathrm{Lip}_M 1$, then

$$| B_n^{k_n}(f; x) - f(x)| \leqslant \frac{M}{2}\frac{k_n}{n}\, x(1 - x) \leqslant \frac{Mk_n}{8n}. \tag{15}$$

Suppose $f \in \mathscr{C}[0, 1]$ and $f \in \mathrm{Lip}_M 1$, then

$$| B_n^{k_n}(f; x) - f(x)| \leqslant \frac{M}{2}\left(\frac{k_n}{n}\right)^{1/2}. \tag{16}$$

For the proof and for further estimates involving other moduli of smoothness, we refer to [5].

We next state a generalization of the Voronovskaja theorem (cf. [3, p. 22]).

THEOREM 2. *Suppose* $f \in \mathscr{C}^{(2)}[0, 1]$ *and* $\lim_{n\to\infty}(k_n/n) = 0$. *Then*

$$\lim_{n\to\infty} \frac{n}{k_n}\{B_n^{k_n}(f; x) - f(x)\} = \frac{1}{2}\, x(1 - x) f''(x) \qquad \textit{uniformly on } [0, 1]. \tag{17}$$

Proof. Since each $B_n^{k_n}$ reproduces linear polynomials, without any loss of generality we restrict ourselves on the subspace

$$\mathscr{C}_0^{(2)}[0, 1] := \{f \in \mathscr{C}^{(2)}[0, 1] \mid f(0) = 0 = f(1)\},$$

which is a normed linear space equipped with the norm $q(f) := \|f''\|_\infty$.

We introduce operators $T_n\colon \mathscr{C}_0^{(2)}[0, 1] \to \mathscr{C}[0, 1]$ $(n \in \mathbb{N}_0)$ defining $T_n f := (n/k_n)\{B_n^{k_n} f - f\}$ for $n \in \mathbb{N}$ and $T_0 f := gf''$, where $g(x) = \frac{1}{2}x(1 - x)$ $(x \in [0, 1])$.

Regarding T_n as linear operators from $\mathscr{C}_0^{(2)}[0, 1]$ with the norm q to $\mathscr{C}[0, 1]$ with the supremum norm, we try to estimate the associated operator norms. Putting $f \in \mathscr{C}_0^{(2)}[0, 1]$ and $\|f''\|_\infty \leqslant 1$, we immediately find $|T_0(f; x)| \leqslant \frac{1}{8}$, and (15) with $M = 1$ yields $|T_n(f; x)| \leqslant \frac{1}{8}$ for $n \in \mathbb{N}$. Hence, the operator norms are uniformly bounded by $\frac{1}{8}$. Due to the theorem of Weierstrass, the polynomials in $\mathscr{C}_0^{(2)}[0, 1]$ form a dense subspace with respect to the norm q. But for polynomials, we have $\lim_{n\to\infty} T_n f = T_0 f$ with respect to the supremum norm on $\mathscr{C}[0, 1]$, on account of (5′). Thus, arguing as in the proof of the theorem of Banach and Steinhaus, we get the assertion $\lim_{n\to\infty} T_n f = T_0 f$ for each $f \in \mathscr{C}_0^{(2)}[0, 1]$. ∎

Finally in an analogous manner, we shall treat the case $\lim_{n\to\infty}(k_n/n) = \infty$. Again applying a Korovkin type theorem of Karlin and Ziegler [1, Theorem 1] and using the test set $\{e_0, e_1, e_2\}$, one obtains for all $f \in \mathscr{C}[0, 1]$

$$\lim_{n\to\infty} B_n^{k_n} f = B_1 f \qquad \text{uniformly on } [0, 1].$$

In order to estimate the quality of the approximation $B_n^{k_n} f - B_1 f$, we again shall apply general quantitative results, derived by the author in [5]. The defects of approximation with the functions of the test set are

$$\tilde{d}_0(x) := B_n^{k_n}(e_0; x) - B_1(e_0; x) = 0,$$

$$\tilde{d}_1(x) := B_n^{k_n}(e_1; x) - B_1(e_1; x) = 0,$$

$$\tilde{d}_2(x) := B_n^{k_n}(e_2; x) - B_1(e_2; x) = \left(1 - \frac{1}{n}\right)^{k_n} x(x - 1).$$

Employing Theorems 6.2 and 6.4 from [5], one obtains the following estimates:

Suppose $f \in \mathscr{C}^{(1)}[0, 1]$ and $f' \in \mathrm{Lip}_M 1$, then

$$|B_n^{k_n}(f; x) - B_1(f; x)| \leqslant \frac{M}{2}\left(1 - \frac{1}{n}\right)^{k_n} x(1 - x) \leqslant \frac{M}{8}\left(1 - \frac{1}{n}\right)^{k_n}. \tag{18}$$

Suppose $f \in \mathscr{C}[0, 1]$ and $f \in \mathrm{Lip}_M 1$, then

$$|B_n^{k_n}(f; x) - B_1(f; x)| \leqslant 2M\left(1 - \frac{1}{n}\right)^{k_n} x(1 - x) \leqslant \frac{M}{2}\left(1 - \frac{1}{n}\right)^{k_n}. \tag{19}$$

To state Theorem 3 parallel with Theorem 2, we need the class

$$\hat{\mathscr{C}}[0, 1] := \{f \in \mathscr{C}[0, 1] \mid f'(0) \text{ and } f'(1) \text{ exist}\}.$$

THEOREM 3. *Suppose* $f \in \hat{\mathscr{C}}[0, 1]$ *and* $\lim_{n\to\infty}(k_n/n) = \infty$. *Then*

$$\lim_{n\to\infty}\left(1 - \frac{1}{n}\right)^{-k_n}\{B_n^{k_n}(f; x) - B_1(f; x)\}$$
$$= 6x(1 - x)\left\{\int_0^1 f(t)\, dt - \frac{f(0) + f(1)}{2}\right\} \qquad \textit{uniformly on } [0, 1]. \tag{20}$$

Proof. Again we may restrict ourselves to the subspace

$$\hat{\mathscr{C}}_0[0, 1] := \{f \in \hat{\mathscr{C}}[0, 1] \mid f(0) = 0 = f(1)\}.$$

We must show that for all $f \in \hat{\mathscr{C}}_0[0, 1]$

$$\lim_{n\to\infty}\left(1 - \frac{1}{n}\right)^{-k_n} B_n^{k_n}(f; x) = 6x(1 - x)\int_0^1 f(t)\, dt \qquad \text{uniformly on } [0, 1].$$

To each $f \in \hat{\mathscr{C}}_0[0, 1]$, there corresponds the function $\hat{f}$ defined by

$$\begin{aligned} \hat{f}(t) &:= f'(0) && \text{for} \quad t = 0,\\ &:= \frac{f(t)}{t(1 - t)} && \text{for} \quad t \in (0, 1),\\ &:= -f'(1) && \text{for} \quad t = 1. \end{aligned}$$

Since $f(0) = 0 = f(1)$ and f is differentiable at 0 and 1, $\hat{f}$ is continuous on $[0, 1]$, and therefore $p(f) := \|\hat{f}\|_\infty$ exists, which is readily proved to be a norm on the space $\hat{\mathscr{C}}_0[0, 1]$. We introduce operators T_n: $\hat{\mathscr{C}}_0[0, 1] \to \mathscr{C}[0, 1]$ $(n \in \mathbb{N}_0, n \neq 1)$ defining

$$T_n f := \left(1 - \frac{1}{n}\right)^{-k_n} B_n^{k_n} f \qquad \text{for} \quad n \geqslant 2$$

and

$$T_0 f := \left(\int_0^1 f(t)\, dt\right) g,$$

where $g(x) = 6x(1 - x)$ $(x \in [0, 1])$. Regarding T_n as linear operators from $\hat{\mathscr{C}}_0[0, 1]$ with the norm p to $\mathscr{C}[0, 1]$ with the supremum norm, we try to estimate the associated operator norms. Putting $f \in \hat{\mathscr{C}}_0[0, 1]$ and $\|\hat{f}\|_\infty \leqslant 1$, we have $|f(t)| \leqslant t(1 - t)$ for all $t \in [0, 1]$. Applying the positive operator $B_n^{k_n}$ yields

$$|B_n^{k_n}(f; x)| \leqslant \left(1 - \frac{1}{n}\right)^{k_n} x(1 - x) \qquad \text{for all} \quad x \in [0, 1]$$

and therefore $\|T_n f\|_\infty \leqslant \frac{1}{4}$. Also we have $\|T_0 f\|_\infty \leqslant \frac{1}{4}$.

Thus the operator norms are uniformly bounded by $\frac{1}{4}$. Now arguing as in the proof of Theorem 2, we obtain the assertion. ∎

We note that for $f \in \mathscr{C}^{(2)}[0, 1]$ (20) can be written in the form

$$\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^{-k_n}\{B_n^{k_n}(f; x) - B_1(f; x)\}$$
$$= -\frac{1}{2}x(1-x)\int_0^1 6t(1-t)f''(t)\,dt. \tag{21}$$

Since $\int_0^1 6t(1-t)\,dt = 1$, the integral in (21) is a weighted mean of the second derivative.

To conclude this paper, we make a remark concerning saturation.

For a function $f \in \mathscr{C}^{(2)}[0, 1]$ in the case $\lim_{n\to\infty}(k_n/n) = 0$,

$$\lim_{n\to\infty}\frac{n}{k_n}\{B_n^{k_n}(f; x) - f(x)\} = 0$$

entails $f \in \mathscr{P}_1$ and $B_n^{k_n}f = f$ for all $n \in \mathbb{N}$, by virtue of Theorem 2. In contrast to this, let $f \in \mathscr{C}[0, 1]$ and $(k_n)_{n\in\mathbb{N}}$ be a sequence with $\lim_{n\to\infty}(k_n/n) = \infty$. Then

$$\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^{-k_n}\{B_n^{k_n}(f; x) - B_1(f; x)\} = 0$$

entails only $\int_0^1 f(t)\,dt = \frac{1}{2}(f(0) + f(1))$, and the example of the function $f(x) = x(x-\frac{1}{2})(x-1)$ shows that $B_n^{k_n}f \neq B_1 f$ for infinitely many n is still possible.

Remarks

1. Theorem 3 is contained in the dissertation [5] of the author.

2. H. J. Rausch, Dortmund, has independently found the polynomials p_j to be an orthogonal set with respect to some inner product.

3. Similar results concerning Kantorovič operators will be published in a forthcoming paper.

References

1. S. Karlin and Z. Ziegler, Iteration of positive approximation operators, *J. Approximation Theory* **3** (1970), 310–339.
2. R. P. Kelisky and T. J. Rivlin, Iterates of Bernstein polynomials, *Pacific J. Math.* **21** (1967), 511–520.

3. G. G. LORENTZ, "Bernstein Polynomials," Univ. of Toronto Press, Toronto, 1953.
4. C. A. MICCHELLI, "Saturation Classes and Iterates of Operators," Ph. D. thesis, Stanford University, 1969.
5. J. NAGEL, "Sätze Korovkinschen Typs für die Approximation linearer positiver Operatoıen," Dissertation, Universität Essen, 1978.
6. R. SCHNABL, Über gleichmäßige Approximation durch positive lineare Operatoren, Proc. Int. Conf. on Constr. Function Theory, Varna, 1970, pp. 287–296.